MODEL CATEGORIES

3/31/21

- Historical (why?)
- definitions
- axioms of model cats.
- examples (Top, Ch(R), ..)
- properties
- homotopies.

0. Why?.

Model categories were introduced by Quillen - Homotopical Algebra (67)

They allow us to "do homotopy" in other categories.

Typically in homotopy theory, we study notions of equivalence. specifically weakening equivalence.

ex) Top < ob: topological spaces; mor: continuous maps.

equivalence (isomorphisms)

homeomorphisms

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

s.t. $f\circ g = id_Y$
$g\circ f = id_X$.

weaken

weak equivalence:

homotopy equivalence. ⇝ weak hom. eq..

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

$f\circ g \sim id_Y$
$g\circ f \sim id_X$

ex) Ch(R) < ob: chain complexes of R-modules; mor: chain homomorphisms.

(R ← ring)

$A_* \in ob(Ch(R))$
is $\{A_k\}_{k\geq 0}$ w/ maps $A_k \xrightarrow{d_k} A_{k-1}$

R-modules

$$\begin{tikzcd}
A_*: \cdots \arrow[r] & A_k \arrow[r, "d_k"] \arrow[d, "f_k"] & A_{k-1} \arrow[r] \arrow[d, "f_{k-1}"] & \cdots \\
B_*: \cdots \arrow[r] & B_k \arrow[r, "d_k"] & B_{k-1} \arrow[r] & \cdots
\end{tikzcd}$$

$f\downarrow$ .. ..

equivalence:
if all $f_k: A_k \to B_k$ are module-iso's.
"chain isomorphisms"

weak

weak equiv:

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(A_*) \arrow[r] \arrow[d] & H_{k-1}(A_*) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_k(B_*) \arrow[r] & H_{k-1}(B_*) \arrow[r] & \cdots
\end{tikzcd}$$

"quasi iso's"

these are isomorphisms

That is, from a category $\mathcal{M}$, with a class of morphisms called "weak equivalences" ($\mathcal{W}$) we want to form a new category called the homotopy category

$$ho(\mathcal{M}) := \mathcal{M}[\mathcal{W}^{-1}]$$

formally inverting all weak equivalences.

In cat. theoretic language, we want a functor

$$Q : \mathcal{M} \longrightarrow \mathcal{M}[\mathcal{W}^{-1}]$$

ob: $A \longmapsto A$

mor: $f \longmapsto Qf$

if $f \in \mathcal{W}$, Qf is an iso. in $\mathcal{M}[\mathcal{W}^{-1}]$.

and $\mathcal{M}[\mathcal{W}^{-1}]$ is universal w/ this property.

ie.
$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{N} \\ \mathcal{M}[\mathcal{W}^{-1}] \arrow[ur, dotted, "\exists !"'] & \end{tikzcd}$$
all weak equiv's are iso's.

The problem is $\mathrm{Hom}_{\mathcal{M}[\mathcal{W}^{-1}]}(X, Y)$ might not be a set.

ie. $\mathcal{M}[\mathcal{W}^{-1}]$ is not "locally small".

Even when it is, it's pretty difficult to use. There is a concrete construction involves "zig-zags", but it's difficult to use, prove things in.

Solution! : Model categories.

Definitions:

$\boxed{\text{retracts}}$: An object $X \in ob(\mathcal{M})$ is a retract of another object $Y \in ob(\mathcal{M})$ if there exist morphisms i, r :

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right, "id_X"'] & Y \arrow[r, "r"] & X \end{tikzcd} \quad \text{s.t. } r \circ i = id_X .$$

ex) In particular, consider the category $\mathrm{Hom}(\mathcal{M})$ ob: morphisms in $\mathcal{M}$; morphisms are pairs of morphisms

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

In this case, a retract looks like a diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "id"] & \bullet \arrow[r] \arrow[d, "g"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] \arrow[rr, bend right, "id"'] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

lifting properties.

Given a square:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, "h"] & Y \end{tikzcd}$$

a $\boxed{\text{lift}}$ is a map $h: B \to X$ making this commute.

Fixing i, p, if a lift exists for all squares of this type, then we say that: i has left-lifting property (LLP) w.r.t. p
p has right " " (RLP) w.r.t. i.

aka. $i \boxslash p$

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model categories.

A $\boxed{\text{model category}}$ $\mathcal{M}$ is a category w/ 3 distinguished classes of morphisms:

(i) $\mathcal{W}$: weak equivalences ($\xrightarrow{\sim}$)
(ii) $\mathcal{F}$: fibrations ($\twoheadrightarrow$)
(iii) $\mathcal{C}$: cofibrations ($\hookrightarrow$, $\rightarrowtail$)

... these mix a little bit..
$\overset{\sim}{\twoheadrightarrow}$ acyclic fibration
$\overset{\sim}{\rightarrowtail}$ acyclic cofibration

satisfying the axioms:

MC1: $\mathcal{M}$ is bicomplete (ie. has all small limits/colimits). — $\mathcal{M}$ has an initial ob. $\emptyset$, $\mathcal{M}$ " " final ob. $*$

MC2-5: $\mathcal{M}$ has a "model structure"

MC2: (2-out-of-3 property): Given two morphisms, $f, g \in \mathrm{Mor}(\mathcal{M})$
$\{f, g, g \circ f\}$
If two of these are weak equivalences, then so is the third.

MC3: (retracts): All classes of morphisms ($\mathcal{W}, \mathcal{C}, \mathcal{F}$) are closed under retracts.

MC4: (lifting): Given a square:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

a lift exists if either: (i is an acyclic cofibration) & (p is a fibration)

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, tail, "\sim"'] & \bullet \arrow[d, two heads] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

or: (i is a cofibration) & (p is an acyclic fib)

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, tail, "i"'] & \bullet \arrow[d, two heads, "p", "\sim"'] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

MC5: (factorization): Any morphism in $\mathcal{M}$ can be factored as:

$$\begin{tikzcd} & \bullet \arrow[dr, two heads] & \\ \bullet \arrow[ur, tail, "\sim"] \arrow[rr, "f"] \arrow[dr, tail] & & \bullet \\ & \bullet \arrow[ur, two heads, "\sim"'] & \end{tikzcd}$$

$X \xrightarrow{f} Y$

Examples:

ex). Top (Quillen model structure)

(i) $\mathcal{W}$: weak homotopy equivalences — $\pi_n(X) \xrightarrow{f_*} \pi_n(Y)$ is an iso. of groups for all n

(ii) $\mathcal{F}$: Serre fibrations:

D^2

$$\begin{tikzcd} D^n \arrow[r] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\ D^n \times I \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

f is a Serre fibration if it has RLP w.r.t. i_0 for all $n \geq 0$.

(iii) $\mathcal{C}$: retracts of maps $\underbrace{X \to Y}_{\text{attaching "n-cells"}}$

$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d, "f"'] & X \arrow[r, "id"] \arrow[d, "\text{(attach cells)}"] & X \arrow[d, "f"] \\ Y \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

$\Rightarrow$ f is a cofibration.

ex) Top again (Strøm model structure) — "The homotopy category is a homotopy category."

$\mathcal{W}$: homotopy equivalences

$\mathcal{F}$: Hurewicz fibrations

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\ A \times I \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

f has RLP w.r.t. i_0.

ex) $Ch(R)$:

$\mathcal{W}$: quasi-isomorphisms

$\mathcal{F}$: $f: A_* \to B_*$ s.t. $f_k: A_k \to B_k$ is surjective for all $k \geq 1$.

$\mathcal{C}$: $f_k: A_k \to B_k$ is injective w/ projective cokernel for all $k \geq 0$.

ex) $\mathcal{M}^{op}$:

$\mathcal{W}$: f^{op} where f is a weak eq. in $\mathcal{M}$.

$\mathcal{F}$: f^{op} where f is a cofibration in $\mathcal{M}$

$\mathcal{C}$: f^{op} where f " " fibration in $\mathcal{M}$

<u>Rk</u>: This makes <u>duality</u> apparent: any statement about a model category can be dualized by:
- reversing all arrows
- switching fibrations & cofibrations.

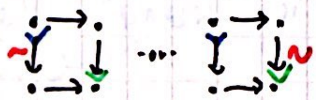
ex) For any bicomplete category $\mathcal{M}$, we can define a model structure by:

- $\mathcal{W}$: isomorphisms in $\mathcal{M}$
- $\mathcal{C}$, $\mathcal{F}$: all morphisms in $\mathcal{M}$

ex)
- $\mathcal{W}$: every map
- $\mathcal{F}$: isomorphisms
- $\mathcal{C}$: (every map?)

ex) Sets has nine model structures.

Properties of Model Categories.

Remember MC4 (lifting)  a lift exists if 

Prop: Acyclic cofibrations are precisely the maps w/ LLP w.r.t. all fibrations.
Cofibrations \ \ LLP w.r.t. all acyclic fibrations.

Pf: (Dwyer + Spalinski).

Rk · This means that model cats are in some sense overdetermined.
ie. if you are describing a model structure, you need only define (weak equivalences) + (either: cofibrations or fibrations).

Rk · Checking the axioms is a pain!

Homotopies. Recall in Top, homotopies look like maps out of a "cylinder object".
ie. $f, g : X \rightrightarrows Y$, we say that f is homotopic to g ($f \sim g$) if there is a map s.t.

$$H : X \times I \longrightarrow Y$$
$$H(-,0) = f$$
$$H(-,1) = g$$

"left homotopy"

Alternatively we can think of a homotopy as a map into a "path object"
ie. $f \sim g$ if $H : X \to Y^I = \mathrm{Maps}(I, Y) \rightrightarrows Y$

these compose to f, g.

"right homotopy"

There is a generalization of both $X \times I$ & Y^I.
They are called ⊛

$$\begin{array}{ccc} X\times I & & Y^I \\ \updownarrow & & \updownarrow \\ \mathrm{Cyl}(X) & & \mathrm{Path}(Y) \end{array}$$

These serve the purpose of $X\times I$ & Y^I in a general model category.

You can generalize notions of L- & R- homotopies.
If two maps, f, g are both left- & right-homotopic, we say that they are <u>homotopic</u>.

<u>The kicker</u>: $f, g : X \rightrightarrows Y$

(i) if X is "cofibrant" ($\emptyset \rightarrowtail X$), then $f \overset{l}{\sim} g \Rightarrow f \overset{r}{\sim} g \Rightarrow f \sim g$
(ii) if Y is "fibrant" ($Y \twoheadrightarrow *$) then $f \overset{r}{\sim} g \Rightarrow f \overset{l}{\sim} g \Rightarrow f \sim g$.

} if both $\Rightarrow$ "bifibrant"

If both X is cofibrant & Y is fibrant, then $\sim$ defines an equivalence relation on $\mathrm{Hom}_{\mathcal{M}}(X, Y)$.

We can ~~now define~~ find for any object $X \in \mathcal{M}$, a $\boxed{\text{cofibrant replacement}}$

$$\begin{tikzcd} \emptyset \arrow[r, tail] \arrow[rr, bend right] & QX \arrow[r, two heads, "\sim"] & X \end{tikzcd}$$

similarly a $\boxed{\text{fibrant replacement}}$:

$$\begin{tikzcd} X \arrow[r, tail, "\sim"'] \arrow[rr, bend left] & RX \arrow[r, two heads] & * \end{tikzcd}$$

} ⊛

Now we can define $\mathrm{ho}(\mathcal{M})$ as a functor $\gamma : \mathcal{M} \to \mathrm{ho}(\mathcal{M})$

$$\text{obj} : X \longmapsto RQX$$
$$\text{mor} : \underset{\substack{\mathrm{ho}(\mathcal{M}) \\ \shortparallel \\ [X,Y]}}{\mathrm{Hom}}(X, Y) := \mathrm{Hom}_{\mathcal{M}}(RQX, RQY) / \sim$$

$f \sim g$ iff they are homotopic.

⊛ This does what we want.

References.

- Dwyer-Spalinski – "Homotopy theories and model categories".
- Hovey - "Model categories"
- Hirschhorn - "Model Categories & their localizations"

} on "Doug's archive".

- May, Ponto - "More Concise Alg. Topology" (for the def. using weak factorization systems)
- ~~Model I~~ Balchin. "Model Categories by Example"
  bifibrant.com/bifibrant.html.

} yt. lectures + lectures notes.

(forthcoming) A Handbook of Model Categories.